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Steady cooling and global overheating processes in a hazardous reactor☆

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ABSTRACT

The non-linear problem of the seepage of a gas in a weakly conducting porous medium, which simulates the overheating of a reactor, is investigated. An analytical calculation of the critical values of the similarity parameter of the problem, which determine the condition for steady cooling of a system that is open to the atmosphere and the condition for global overheating, is carried out. Estimates of the above-mentioned values of the similarity parameter are obtained in the three-dimensional case.

1. Introduction

Measurements in the neighbourhood of an obstruction in a hazardous reactor and the results of physical experiments on samples have shown that, of the different models for the evolution of the active zone of a reactor, only the seepage cooling model,[1] that is, the model of the seepage of a gas through a self-warming porous medium in a gravitational force field, proved to be sufficiently practicable. This model enables one to explain why fuel element fragments have not burnt through the base of a reactor.

A reactor can be both in a fairly stable state as well as in a unsafe state with the prospect of overheating. The problem of the subsequent analysis involves diagnosis and distinguishing between the above two situations as a continuation of the preceding investigations.[1,2] To begin with, it is a question of establishing a numerical criterion which determines the conditions for the existence of a steady cooling process in a system which is open to the atmosphere.

Two important critical values M_* (Ref. 1) and M_{**} (Ref. 2) of the parameter M have been introduced characterizing the level of the distribution $Q(x, y, z)$ of the heat sources in the reactor

$$Q(x, y, z) = M^2 q(x, y, z); \qquad \int_V q(x, y, z)dV = 1 \tag{1.1}$$

where V is the volume of the reactor and M_* is the value of the parameter M at which the pressure gradient of the gas (grad$P$) vanishes at a certain point when $z = 0$ (on the bottom of the reactor). This value corresponds to the point where one process changes into the other, that is, from thermal equilibrium to local thermal overheating. The second critical constant, $M = M_*$, is the minimum value of the parameter M, at which there is a point on each trajectory of the field grad$P$ at which grad $P = 0$. There is a significant increase in the reactor temperature at this value and global overheating of the system occurs.

Estimates of the constant M_* have been obtained earlier[1] for special cases of a one-dimensional obstruction. An increase and decrease in the magnitude of M_* as a function of a change in the permeability of the obstruction has been established. The behaviour of the basic physical quantities when M tends to M_* has been studied: the entropy of the gas in the obstruction increases without limit, the pressure of the gas and the rate of seepage remain finite, and the total flow rate of the gas through a horizontal section of the obstruction is equal to zero.

The principal aim then is to make an analytical estimate of the critical constants M_* and M_{**} in the general three-dimensional case. Obviously, when $M < M_*$, the reactor is stable.

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2. The model seepage problem

We will now formulate the model problem which is being studied. Suppose the volume V is bounded by a certain complex lateral surface S, a bottom $z=0$ and a cover $z=H$. The three-dimensional gas flow in a weakly conducting porous medium filling the volume V is considered.

First of all, the question arises as to the correctness of the use of the Darcy seepage model. An obstruction in the compartment of the emergency unit may contain "flaws" channels of large diameter with a low hydraulic resistance and a high flow rate of the gas through these channels. However, special experiments on physical samples which have been described earlier[1] and direct measurements of the temperature level of the gas on the surface of an obstruction establish that there are no such flaws and justify the possibility of describing the seepage process in question using the Darcy model.

We next make the standard change to dimensionless physical characteristics of the obstruction medium by introducing the ratios of the quantities to their scale values.

The gas flow is described by Darcy's equation, the equation of continuity and the energy equation

$$\operatorname{grad} P = -\frac{1}{T}\mathbf{e}_z - \lambda\mathbf{v}, \quad \operatorname{div}(T^{1}\upsilon) = 0, \quad (\mathbf{v}, \operatorname{grad}\ln T) - \varepsilon\Delta T = M^2 q, \quad \varepsilon \ll 1 \tag{2.1}$$

Eqs. (2.1) define the relations between the dimensionless quantities. Here T is the temperature, $\mathbf{v}$ is the seepage rate of the gas and $\lambda=\beta/k$ (β is the viscosity and k is the permeability of the medium). The low value of the dimensionless parameter ε is due to the low thermal conductivity of the gas in the obstruction. The differential operators div and grad and the Laplacian operator are three-dimensional. The gas density ρ is eliminated from the first two Eqs. of (2.1) using the equation of state of a perfect gas

$$P=\rho T,$$

which is assumed to be satisfied to a sufficient approximation. The quantities λ and Q (the distribution of heat sources (1.1)) are assumed to be specified.

It is assumed in connection with Darcy's equation that the pressure within the obstruction is not too low and, on the other hand, the rate of seepage is small, which leads to a linear law. The gas viscosity, β, is practically independent of the pressure and, although β is temperature dependent, it only influences the main effects in a corrective manner.

The energy equation is written on the assumptions that the dissipative function of the system is small and that the temperature dependence of the thermal conductivity can be neglected. In this equation, the entropy is expressed in terms of $\ln T$. The equations of motion (2.1) simulate the dynamics of the temperature, pressure and convective gas flows under the cover of the reactor. We will assume that the heat sources do not move during the seepage process and that there are no chemical reactions and melting of the substances in the volume V.

The boundary conditions on the bottom and the cover of the reactor have the form

$$\upsilon_z = \begin{cases} \alpha_1(P_a^+ - P), & z = 0 \\ \alpha_2(P - P_a^-), & z = H \end{cases} \tag{2.2}$$

Here α_1 and α_2 are the conductivities, and P_a^+ and P_a^- are the values of the atmospheric pressure on the bottom and the cover. For simplicity, we will henceforth assume that $\alpha_1=\alpha_2=\alpha$. It is obvious that

$$P_a^+ - P_a^- = \bar{\rho} g H \tag{2.3}$$

where $\bar{\rho}$ is the mean density of the medium of the obstruction.

The temperature on the bottom is assumed to be constant:

$$z=0: \quad T=T_0 \tag{2.4}$$

and the condition

$$z=H: \quad \varepsilon_0 \partial T/\partial z = 0, \quad \varepsilon_0 \ll 1 \tag{2.5}$$

is approximately satisfied on the cover. As a consequence of the smallness of the thermal conductivity of air, ε_0, it is always approximately satisfied.

Experimental investigations in the lower part of the reactor compartment show that atmospheric air enters through the impermeability condition apertures which are present in the bottom of the reactor, maintaining the bottom at a constant temperature, which corresponds to boundary condition (2.4). Condition (2.5) is an analogue of the Dankwerts condition for chemical reactors. The condition of the continuity of heat fluxes and the impermeability condition are satisfied on the side surface of the reactor. The effect of the boundary conditions on the side surface depends very much on whether or not there are "pockets" in the side wall. The existence of such pockets means that, for certain sections of the reactor with planes of the form $z=\text{const}$, the reactor zone can turn out to be multiply connected.

3. Transformation of boundary value problem (2.1)–(2.5)

Using the first Eq. of (2.1), we eliminate the seepage rate from the system of equations. The remaining two equations take the form

$$\operatorname{div}(T^{-1}\operatorname{grad} P) + \frac{\partial}{\partial z}T^{-2} = 0 \tag{3.1}$$

$$(\operatorname{grad} P, \operatorname{grad} T) + T^{-1}\partial T/\partial z + \lambda\varepsilon T\Delta T = -\lambda q M^2 T \tag{3.2}$$

Eq. (3.1) is written on the natural assumption that the permeability k varies much more slowly than the quantities T and P.

We transform boundary conditions (2.2), using the first Eq. of (2.1) and equality (2.3):

$$z=0:\quad \partial P/\partial z + T^{-1} - \lambda\alpha P = -\lambda\alpha P_a^{+}, \qquad z=H:\quad \partial P/\partial z + T^{-1} + \lambda\alpha P = \lambda\alpha P_a^{-} \tag{3.3}$$

The remaining boundary conditions for the quantities $\partial P/\partial n$, $1/T$ and $\partial T/\partial n$ are already partially defined by equalities (2.4) and (2.5), and the condition of continuity for the heat fluxes and the impermeability condition already mentioned above have the form

$$\partial P/\partial n + T^{1}(e_z, n) = 0, \quad \partial T/\partial n = 0 \text{ на } S \tag{3.4}$$

In the case considered here $M \le M_*$, the small quantity $1/T$ as well as the small parameter ε appear in system of Eqs. (3.1), (3.2). The smallness of the quantity $1/T$ in the upper part of the reactor is also caused by the fact that fragments of the fuel elements within the reactor only heat those layers of the solid substance located above these sources. Important particular examples[1] show that the parameter M_* is finite.

We will now carry out a more exact comparison of the small quantities. An analysis of Eq. (3.12) shows that the quantity $|\mathrm{grad}P|$ is of the same order of smallness as $1/T$. The predominance of one of the terms of this equation over the other would obviously lead to trivial solutions for $1/T$ or for $\mathrm{grad}P$. On the basis of Eqs. (3.1), (3.2), it is now easy to establish that the derivatives $\partial T/\partial x$ and $\partial T/\partial z$ are large and of the order of T^2, and that the second derivatives of the temperature are of the order of T^3. It therefore follows from Eq. (3.2) that the small parameter ε is either of the order of $1/T^3$ or is an even smaller quantity and the expression $\varepsilon\Delta T$ is either small or bounded.

We now transform problem (3.1)–(3.4) and rewrite Eq. (3.1) in the form

$$T^{-1}(\mathrm{grad}P, \mathrm{grad}T) - \Delta P + 2T^{-2}\partial T/\partial z = 0$$

We eliminate the scalar product of the pressure and temperature gradients from this equation and Eq. (3.2). The resulting equation is

$$\Delta P + \partial T^{-1}/\partial z + \varepsilon\lambda\Delta T = -\lambda q M^2 \tag{3.5}$$

and we integrate over the volume V of the reactor using Gauss's formula. After grouping the terms, we find

$$\int\limits_S \left(\frac{\partial P}{\partial n} + \varepsilon\lambda\frac{\partial T}{\partial n}\right)dS + \int\limits_{S_t}\left(\frac{\partial P}{\partial z} + \varepsilon\lambda\frac{\partial T}{\partial z} + T^{-1}\right)dxdy - \int\limits_{S_b}\left(\frac{\partial P}{\partial z} + \varepsilon\lambda\frac{\partial T}{\partial z} + T^{-1}\right)dxdy + \lambda M^2 = 0 \tag{3.6}$$

Here S_b is the surface of the bottom ($z=0$), S_t is the surface of the cover ($z=H$) and S is the lateral surface of the reactor.

We use boundary conditions (3.3) and (3.4). Then,

$$M^2 = \int\limits_{S_b}[\alpha(P-P_a^{+}) + \varepsilon\partial T/\partial z]dxdy + \int\limits_S T^{-1}(\mathbf{n}, \mathbf{e}_z)dS + \int\limits_{S_t}[\alpha(P-P_a^{-}) + \varepsilon\partial T/\partial z]dxdy \tag{3.7}$$

4. The temperature and pressure in the neighbourhood of the bottom and the cover of the reactor

Formula (3.7) contains integrals of the derivative $\varepsilon\partial T/\partial z$ over the surfaces S_b and S_t. We now make the contribution of the corresponding terms more precise. The parameter ε is small but the derivative $\partial T/\partial z$ can be large.

We construct approximate solutions of Eq. (3.1) close to the bottom of the reactor in the form of sections of Taylor series

$$T = T_0 + (\partial T/\partial z)_0 z + \ldots, \qquad P = P_0 + (\partial P/\partial z)_0 z + \ldots \tag{4.1}$$

Here T_0 (see (2.4)) is a constant, the magnitudes of P_0 and $(\partial T/\partial z)_0$ can depend on the horizontal variables x and y, and, at the same time, there is a relation between them by virtue of the first condition of (3.3).

Substituting expansions (4.1) into Eq. (3.1) and equating the coefficients of the zeroth and first powers of z we obtain an enormous system of partial differential equations with respect to the functions $P_0(x, y)$ and $(\partial T/\partial z)_0(x, y)$. However, it turns out that this system of equations can be converted into two separate equations for each of the functions. As a result, it is possible to establish that P_0 and $(\partial T/\partial z)_0$ are independent of x and y, and

$$P_0 = \mathrm{const}, \qquad (\partial T/\partial z)_0 = 0 \tag{4.2}$$

The use of sections of series in powers of $(H - z)$, similar to (4.1), enables us to carry out the same calculations in the neighbourhood of the cover and to establish that

$$z = H:\quad \partial T/\partial z = 0 \tag{4.3}$$

In particular, this means that condition (2.5) is not approximately but exactly satisfied.

Note that the use of expansions of the type (4.1) enables us to obtain estimates of the quantities being studied not only when $z \to 0$ but, also, for small but finite values of z (see Ref. 3, Chapter 6).

5. Estimate for M_*

Taking account of the second equality of (4.2) and equality (4.3), we now have

$$M^2 = \alpha\int\limits_{S_b}(P-P_a^{+}) + \alpha\int\limits_{S_t}(P-P_a^{-})dxdy + \int\limits_S T^{-1}(\mathbf{n}, \mathbf{e}_z)dS \tag{5.1}$$

The sum of the integrals over the bottom and the cover in relation (5.1) is always positive by virtue of equality (2.3) and the equation of state of the gas. If there are no pockets and the lateral surface is close to circular cylindrical, the contribution of the integral over the lateral surface S is negligibly small owing to the simultaneous smallness of the factors $(\mathbf{n}, \mathbf{e}_z)$ and $1/T$. The sign of the scalar product $(\mathbf{n}, \mathbf{e}_z)$ changes, depending on the slope of the boundary S, which additionally reduces the contribution from the integral over S. Hence, when there are no pockets, formula (5.1) gives an explicit positive estimate of the critical constant M_*. Hence, when $M^2 < M_*^2$, according to what has been said above, the reactor is in a stable state.

We will now consider the effect of pockets on the right-hand side of (5.1). The factor $(\mathbf{n}, \mathbf{e}_z)$ obviously takes positive values in the lower part of each pocket and negative values in the upper part of it. Simultaneously, the magnitude of T becomes larger as the z coordinate increases. This is due to the heating of the upper layers by fuel element fragments mentioned above and, also, the air cooling of the lower part of the reactor. The negative contribution of the lower parts of the pockets therefore predominates over the positive contribution of the upper parts. In the final analyses, the integral over S is negative on account of the existence of pockets, although it has an extremely small modulus as a consequence of the smallness of the quantity $1/T$.

In the axi symmetric case, the contribution from the pockets can be estimated exactly. In r, z coordinates the integral over S from (5.1) transforms into a contour integral.

Suppose the pocket is bounded by an "almost" closed smooth contour encompassing an area D. Then, by Green's formula, the contribution of the pocket is determined by the integral $\int_D (\mathbf{e}_z,\ \mathrm{rot}T^{-1}\mathbf{e}_r)d\sigma$. As a consequence of the axial symmetry, $\mathrm{rot}(\mathbf{e}_r/T) = 0$ and the integral over D is equal to zero.

Hence, in the axi symmetric case, the presence of pockets has no effect on the magnitude of M_* and, in the general three-dimensional case, pockets either do not change the magnitude of the critical constant or only slightly reduce it.

Finally, because of the insignificance of the contribution from the integral over the lateral surface S, we obtain the following explicit estimate

$$M_*^2 \cong \alpha\int\limits_{S_b}(P - P_a^+)dxdy + \alpha\int\limits_{S_t}(P - P_a^-)dxdy \tag{5.2}$$

6. The high-temperature case. Estimate for M_{**}

We will now consider the situation when the reactor is close to a state of overheating. As before, we assume that the fuel elements in the reactor do not move and that Darcy's law holds.

If the temperature T is sufficiently large, the term $T^{-1}\mathbf{e}_z$ in Darcy's Eq. (2.1) can be neglected. Eqs. (3.1) and (3.5) then take the form

$$T^{-1}(\mathrm{grad}P, \mathrm{grad}T) + \lambda\varepsilon\Delta T = -\lambda q M^2, \quad \mathrm{div}(\mathrm{grad}P) + \lambda\varepsilon\Delta T = -\lambda q M^2 \tag{6.1}$$

When $M \geq M_{**}$, the term $\lambda\varepsilon\Delta T$ in the first Eq. of (6.1) can no longer be neglected. This means that, in this case,

$$T \sim \tau/\varepsilon \tag{6.2}$$

where τ is a finite quantity.

In neglecting the term $\lambda\varepsilon\Delta T$ in the second Eq. of (6.1), we conclude that, according to the definition of M_{**}, the vector field $\mathrm{grad}P$ has a zero on each trajectory of this field. At the same time, the solution of the first Eq. of (6.1) when $\varepsilon = 0$ has a singularity on each trajectory of the field $\mathrm{grad}P$. Hence, a high temperature with the asymptotic behaviour (6.2) corresponds to the condition $M \geq M_{**}$. This precisely means that a situation in which there is global overheating of the system has occurred.

If, however, $M < M_{**}$, then asymptotics (6.2) only hold in the domain covered by the trajectories on which $\mathrm{grad}P \neq 0$. This means that, when $M < M_{**}$, the temperature is no longer of the order of (6.2) everywhere.

We will now define the behaviour of the temperature and pressure in the high-temperature case more precisely. Instead of Eqs. (3.1) and (3.5), we obtain

$$\mathrm{div}(T^{-1}\mathrm{grad}P) = 0, \qquad \Delta P + \varepsilon\lambda\Delta T = -\lambda q M^2 \tag{6.3}$$

In accordance with asymptotics (6.2), we will seek a solution of these equations in the form

$$T = \tau/\varepsilon + O(1), \qquad P = p + O(\varepsilon)$$

which leads to the system of equations

$$\mathrm{div}(\tau^{-1}\mathrm{grad}p) = 0, \qquad \Delta(p + \lambda\tau) = -\lambda q M^2 \tag{6.4}$$

The boundary conditions for τ and p acquire the form

$$z = 0\text{: } \tau = 0,\ \partial p/\partial z - \lambda\alpha p = -\lambda\alpha P_a^+; \qquad z = H\text{: } \partial\tau/\partial z = 0,\ \partial p/\partial z + \lambda\alpha p = \lambda\alpha P_a^-$$

$$\partial p/\partial n = 0, \quad \partial\tau/\partial n = 0 \text{ на } S \tag{6.5}$$

The subsequent transformations are analogous to the derivation of the estimate for M_* presented above. We integrate the equations over the volume of the reactor using Gauss formula. Integration of the left-hand side of Eq. (6.4) after using the boundary conditions only gives a result which is identically zero when the following relation is satisfied

$$z = 0\text{:} \quad p = P_a^+ \tag{6.6}$$

Integration of the second Eq. of (6.4) leads to the equality

$$\int_{S}(\partial p/\partial n+\lambda\partial\tau/\partial n)dS+\int_{S_t}(\partial p/\partial z+\lambda\partial\tau/\partial z)dxdy+\int_{S_b}(\partial p/\partial z+\lambda\partial\tau/\partial z)dxdy = -\lambda M^2$$

which is simplified by using boundary conditions (6.5) and (6.6) and equality (2.3).

We find

$$\lambda M^2 = \lambda\alpha\bar{\rho}gH\Sigma+\int_{S_b}(\partial p/\partial z+\lambda\partial\tau/\partial z)dxdy \tag{6.7}$$

where Σ is the area of the cover of the reactor.

To evaluate the derivatives on the bottom of the reactor $\partial p/\partial z$ and $\partial\tau/\partial z$, we again use expansions of the type (4.1). The result turns out to be as follows:

$$z=0:\quad \partial p/\partial z=0,\quad \partial\tau/\partial z=0 \tag{6.8}$$

as a consequence of which the integral term in equality (6.7) turns out to be zero.

If the product $H\Sigma$ is put approximately equal to the volume of the reactor, then the resultant formula for M_{**} takes the form

$$M_{**}^2 \cong \alpha\Pi \tag{6.9}$$

where Π is the weight of the whole reactor together with the obstruction. Obviously,

$$M_{**} \geqslant M_{*}$$

7. Conclusion

The non-linear system of Eq. (2.1), together with the corresponding boundary conditions, has been investigated analytically. This system described the basic processes taking place in a hazardous reactor, the bottom of which is in contact with the atmosphere. The problem reduces to the non-linear system of Eqs. (3.1), (3.5) which relates the two main quantities, temperature and pressure.

Integration of the system of equations obtained led to the explicit formula (5.1) for the first critical value M_* of the similarity parameter M of the problem. This critical value corresponds to the point at which there is a changeover of processes, that is, a transition from thermal equilibrium (when $M < M_*$) to local thermal overheating (when $M > M_*$). The temperature and pressure close to the bottom and the cover of the reactor were calculated. The role of pockets originating in the wall of a hazardous reactor was estimated. The calculations and estimates enabled us to simplify formula (5.1).

Attainment of the second critical value M_{**} of the similarity parameter is associated with the transition of the reactor to a state of global overheating. The characteristic indication that this state has been attained is established: the temperature simultaneously becomes a quantity of the order of (6.2). An analysis of the equations for this case leads to estimate (6.9) for the quantity M_{**}.

The main result of this paper is the explicit and fairly simple estimates (5.2) and (6.9) for the critical values of the similarity parameter.

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